Sec 11-8: Binomial Theorem

<u>Recall:</u> A Binomial is simply an expression with two unlike terms.

We're going to look at raising the Binomial $(a+b)$ to consecutive powers and see what <u>patterns</u> may exist.

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a^1b^0 + 1a^0b^1$$

$$(a+b)^2 = 1a^2b^0 + 2a^1b^1 + 1a^0b^2$$

$$(a+b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$

$$(a+b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$$

$$(a+b)^5 = 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$$

The coefficients form a pattern often displayed in a triangular formation. This is known as Pascal's Triangle! Each row begins & ends with 1, and you sum the coeffs to get to next row.

$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	1 5 10 10 5 1
$(a+b)^6$	1 6 15 20 15 6 1
$(a+b)^7$	1 7 21 35 35 21 7 1
$(a+b)^8$	1 8 28 56 70 56 28 8 1
$(a+b)^9$	1 9 36 84 126 126 84 36 9 1
$(a+b)^{10}$	1 10 45 120 210 252 210 120 45 10 1

Expand $(a+b)^7 = 1a^7b^0 + 7a^6b^1 + 21a^5b^2 + 35a^4b^3$
$+ 35a^3b^4 + 21a^2b^5 + 7a^1b^6 + 1a^0b^7$

$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3$
$+ 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

Sometimes we have a product of descending factors, such as:

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

This canbeexpressed using <u>factorial</u> notation. $4! = 4 \cdot 3 \cdot 2 \cdot 1$

This is read as "Four factorial."

On Calc → $\boxed{\text{MATH}}$ → $\boxed{\text{PRB}}$ $\boxed{\#4!}$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

ex.) $\dfrac{5!}{3!\,2!} = \dfrac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot 2 \cdot 1 \cdot \cancel{2} \cdot \cancel{1}} = \dfrac{20}{2} = 10$

p. 699 #5 $8! = 40{,}320$ #6 $\dfrac{13!}{9!} = 17{,}160$

#7 $\dfrac{12!}{2!\,10!} = 66$

Binomial Theorem

Pascal's Triangle gives us coefficients for $(a+b)^n$ for any value of n, where n is a positive integer.

This pattern is summarized by the Binomial Theorem:

$$(a+b)^n = a^n + \frac{n!}{1!(n-1)!}a^{n-1}b^1 + \frac{n!}{2!(n-2)!}a^{n-2}b^2 + \dots b^n$$

$$(a+b)^3 = a^3 + \frac{3\cdot\cancel{2}\cdot\cancel{1}}{1\cdot\cancel{2}\cdot\cancel{1}}a^2b^1 + \frac{3\cdot\cancel{2}\cdot 1}{\cancel{2}\cdot 1\cdot 1}a^1b^2 + \frac{\cancel{3}\cdot\cancel{2}\cdot 1}{\cancel{3}\cdot\cancel{2}\cdot 1\cdot 1}a^0b^3$$

$$a^3 + 3a^2b + 3ab^2 + b^3$$

Ex.) Expand $(\underset{a}{3m} + \underset{b}{d})^{5 \leftarrow n}$

$$(3m)^5 + \frac{5!}{1!(4!)}(3m)^4 d^1 + \frac{5!}{2!(3!)}(3m)^3 d^2$$

$$+ \frac{5!}{3!(2!)}(3m)^2 d^3 + \frac{5!}{4!(1!)}(3m)^1(d^4) +$$

$$\frac{5!}{5!\,0!}(3m)^0 d^5$$ Simplify

$$243m^5 + 5\cdot 81m^4 d + 10\cdot 27m^3 d^2 + 10\cdot 9m^2 d^3$$
$$+ 5\cdot 3md^4 + d^5$$

$$= 243m^5 + 405m^4 d + 270m^3 d^2 + 90m^2 d^3$$
$$+ 15md^4 + d^5.$$

Ex.) $(3x-4y)^4$

$$(3x)^4 + \frac{4!}{1!(3!)}(3x)^3(-4y)^1 + \frac{4!}{2!2!}(3x)^2(-4y)^2$$

$$+ \frac{4!}{3!1!}(3x)^1(-4y)^3 + \frac{4!}{4!0!}(3x)^0(-4y)^4$$

$$= 81x^4 + 4(27x^3)(-4y) + 6 \cdot 9x^2 \cdot 16y^2$$
$$+ 4 \cdot 3x \cdot -64y^3 + 256y^4$$

$$= 81x^4 + -432x^3y + 864x^2y^2$$
$$+ -768xy^3 + 256y^4$$

P. 699 Fly through # 14 - 20
23 - 26

* Use Pascal's Triangle to expand any Binomial.

Ex.) $(x+y)^7$

Pascal $(a+b)^7$: 1, 7, 21, 35, 35, 21, 7, 1

* remember the first term starts with highest exponent and descends↓. The 2nd term starts w/ zero exponent and ascends↑

$1\cdot x^7y^0 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4$

$+ 21x^2y^5 + 7x^1y^6 + 1\cdot x^0y^7$

$$\boxed{= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7}$$

Expand $(3m+d)^5$ Pascal: 1, 5, 10, 10, 5, 1

$= 1\cdot(3m)^5(d)^0 + 5(3m)^4(d)^1 + 10(3m)^3(d)^2$

$+ 10(3m)^2(d)^3 + 5(3m)^1(d)^4 + 1(3m)^0(d)^5$

$= 243m^5 + 405m^4d + 270m^3d^2$

$+ 90m^2d^3 + 15md^4 + d^5$

Ex.) $(2a-3b)^4$ Pascal: 1, 4, 6, 4, 1

$= 1\cdot(2a)^4(-3b)^0 + 4(2a)^3(-3b)^1 + 6(2a)^2(-3b)^2$

$+ 4(2a)^1(-3b)^3 + 1(2a)^0(-3b)^4$

$= 16a^4 + 32a^3(-3b) + 24a^2(9b^2)$

$+ 8a(-27b^3) + 81b^4$

$= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$