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HansenMath™ Pre-calc: 7.7 Determinant of a Square Matrix

Warm-up: Use the formula we learned in the previous section--pictured below--to find the inverse of [A]

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, the inverse given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Formula for inverse of matrix A

$$A^{-1} = \frac{1}{4-(-3)}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \rightarrow \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

The denominator of the fraction part, **ad - bc**, is called the *determinant*. One way to remember this is that it's the negative-slope diagonal product minus the positive-slope diagonal product.

So, the determinant of Matrix [A] above is: 7

The determinant is used in calculus and application problems. We'll look at one or two applications in the next, final section of Chapter 7. But let's just try the algorithm first - you have to know how to turn a screwdriver before you can fix anything!

Example 1a: Find the determinant of the following Matrices:

$$\textbf{b. } B = \begin{array}{c} \begin{matrix} a & b \end{matrix} \\ \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \\ \begin{matrix} c & d \end{matrix} \end{array} \qquad \textbf{c. } C = \begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}$$

$$\det B = 4 - 4 = 0$$

$$\det C = 0 - 3 = -3$$

Example 1b: Use a graphing utility to verify the determinant of exercise C above.

Enter into Matrix [A]. (2ND) Matrix → Edit

(2ND) Matrix → MATH → det[A] = -3

Note: *The determinant of a matrix of order 1 x 1 is defined simply as the sole entry of the matrix*

Ex.) [5] → det is still 5

Minors and Cofactors

To define the determinant of a square matrix of order 3 × 3 or higher, it is helpful to introduce the concepts of **minors** and **cofactors**.

Note: not the "literal" signs

Neg denotes the opposite sign change

Sign Patterns for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3 × 3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4 × 4 matrix

Minors and Cofactors of a Square Matrix

If A is a square matrix, the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is given by

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

* See Video *

Example 2 Finding the Minors and Cofactors of a Matrix

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Minors:

$M_{11} = -1-0 = \boxed{-1}$ $M_{12} = 3-8 = \boxed{-5}$ $M_{13} = 0-(-4) = \boxed{4}$

$M_{21} = 2-0 = \boxed{2}$ $M_{22} = 0-4 = \boxed{-4}$ $M_{23} = 0-8 = \boxed{-8}$

$M_{31} = 4-(-1) = \boxed{5}$ $M_{32} = 0-3 = \boxed{-3}$ $M_{33} = 0-6 = \boxed{-6}$

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change signs → *see chart

Cofactors:

$C_{11} = -1$ $C_{12} = 5$ $C_{13} = 4$

$C_{21} = -2$ $C_{22} = -4$ $C_{23} = 8$

$C_{31} = 5$ $C_{32} = 3$ $C_{33} = -6$

The Determinant of a Square Matrix

The following definition is called *inductive* because it uses determinants of matrices of order $n - 1$ to define determinants of matrices of order n.

Determinant of a Square Matrix

If A is a square matrix (of order 2 × 2 or greater), the determinant of A is the sum of the entries in any row (or column) of A multiplied by their respective cofactors. For instance, expanding along the first row yields

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Applying this definition to find a determinant is called **expanding by cofactors.**

Example 3 The Determinant of a Matrix of Order 3 × 3

Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

* you get to pick the row to work on! And multiply by the element's corresponding Cofactor!

$\det = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13}$

$[A] = 0 \cdot -1 + 2 \cdot 5 + 1 \cdot 4$

$= 0 + 10 + 4 = \boxed{14}$

Triangular Matrices

Evaluating determinants of matrices of order 4 × 4 or higher can be tedious. There is, however, an important exception: the determinant of a **triangular** matrix. A triangular matrix is a square matrix with all zero entries either below or above its main diagonal. A square matrix is **upper triangular** if it has all zero entries below its main diagonal and **lower triangular** if it has all zero entries above its main diagonal. A matrix that is both upper and lower triangular is called **diagonal.** That is, a diagonal matrix is a square matrix in which all entries above and below the main diagonal are zero. * special Cases

Example 4: Upper Triangular

24. $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

Just multiply the Main Diagonal elements together!

$\det = (-3)(11)(2) = -66$

Example 5: Lower Triangular

30. $\begin{vmatrix} 5 & -10 & 1 & 1 \\ 0 & 6 & 3 & 4 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$

$\det = (5)(6)(-2)(-1) = 60$

Example 6: Diagonal

36. $\begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix}$

$\det = (-2)(3)(-1)(2)(-4) = -48$

<u>Assignment</u>: **Page 556: #1-33 EOO. #67-68. Thank you so much ☺**