Name: 3/4/2020

HansenMath™ Pre-calc: 7.6 Inverse of a Square Matrix

Do you remember the definition of the *multiplicative identity property*? It simply means that any time you multiply a number by 1, the result is the original number. In other words, it preserves the original number's identity! For example, 23 x 1 = 23. See, the number 23 gets to keep its identity ☺

With Matrices, the idea is very similar. However, instead of multiplying a matrix by a scalar of 1, you can multiply by what's called an *identity matrix*. This is a square matrix consisting of 1s on the diagonal and zeroes elsewhere. Let's take a look ☺

Example 1: Multiply a Matrix by its identity Matrix

A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ What is A*I?

$$\begin{array}{cc} & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 1+0 & 0+2 \\ 3+0 & 0+4 \end{bmatrix} \end{array} \rightarrow \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{[A]}$$

Next, remember that a *multiplicative inverse* is the number you multiply to another number to get 1.

For example, 8 * $\frac{1}{8}$ = 1

With Matrices, the idea is very similar. If you multiply a matrix A, by its inverse A^{-1}, you get the Identity Matrix.

Example 2: Multiply to get the product, AB.

A = $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ B = $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ $\rightarrow$

$$\begin{array}{cc} & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} \end{array} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Identity}}$$

What must be true of Matrix A and B then? Inverses of each other.

But how do we find an inverse in the first place???

If a matrix A has an inverse, A is called **invertible** (or **nonsingular**); otherwise, A is called **singular**.
A non-square matrix cannot have an inverse. Not all square matrices have inverses. If a matrix does have an inverse, that inverse is unique.

Example 3: Finding the Inverse of a Matrix, take one!

Find the inverse of: $\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$

Finding an Inverse Matrix

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \,\vdots\, I]$.
2. If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \,\vdots\, I]$. The result will be the matrix $[I \,\vdots\, A^{-1}]$. If this is not possible, A is not invertible.
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

$[A] * [A^{-1}] \quad I$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{11}+4X_{21} & X_{12}+4X_{22} \\ -X_{11}-3X_{21} & -X_{12}-3X_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} X_{11}+4X_{21} &= 1 \\ -X_{11}-3X_{21} &= 0 \end{aligned}\right\} \begin{aligned} X_{21} &= 1 \\ X_{11} &= -3 \end{aligned} \qquad \left.\begin{aligned} X_{12}+4X_{22} &= 0 \\ -X_{12}-3X_{22} &= 1 \end{aligned}\right\} \begin{aligned} X_{22} &= 1 \\ X_{12} &= -4 \end{aligned}$$

Inverse

$$\rightarrow \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

Example 4: Finding the Inverse of a Matrix, take two!

Find the inverse of: $\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$

Calc → 2ND Matrix → [A]

RREF

$$\left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \rightarrow \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

2×4

Inverse

Example 5: Find the Inverse - BUT this time use your calculator!

Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$. ←

Edit Matrix [A]

Inverse button $\boxed{X^{-1}}$

3×3

$$[A]^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

Example 6: Find the Inverse - use your calculator again!

$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$ No Inverse

*** There is also a special, simpler formula for finding the inverse of 2x2 matrices ONLY ***

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$ Formula for inverse of matrix A

Example 7: Use the formula above to find the inverse of the 2x2 matrix.

$$A = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array}.$$

$$A^{-1} = \frac{1}{6-2}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \to \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 1/2 & \frac{3}{4} \end{bmatrix}$$

Example 8: Use an inverse Matrix to solve the system! (An alternative to RREF)

$$\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}$$

Coefficient $\qquad [S] \qquad [B]$

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$[A]$

$$[A]^{-1} * [B] = [S] \to \begin{array}{c} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \end{array} \begin{array}{c} x \\ y \\ z \end{array}$$

Assignment: **Page 547: #1, 3, 5 (Multiply, by hand)**

#11, 13, 15 (by hand, any method from these notes)

#21, 25, 51, 55, 75 (Use calculator. Use any method you know to solve)